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Periodic solutions of impulsive systems with a small delay

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Abstract. For an impulsive system with delay it is proved that if the corresponding system without delay has an isolated ω-periodic solution, then in any neighbourhood of this orbit the system considered also has an ω-periodic solution if the delay is small enough.

1. Introduction

In the mathematical simulation of the evolution of real processes in physics, chemistry, population dynamics, radio engineering, etc which are subject to disturbances of negligible duration in comparison with the total duration of the process, it is often convenient to assume that the disturbances are 'momentary', in the form of impulses. This leads to the investigation of differential equations and systems with discontinuous trajectories, or with impulse effect, called, for the sake of brevity, impulsive differential equations and systems [1, 2].

In many applications it is assumed that the system considered is subject to the causality law, i.e. the future state of the system does not depend on the past states and is determined by the present only. But in the more detailed investigation it often becomes obvious that the causality law is just a first approximation to the real situation, and a more realistic model should involve some of the past states of the system. Moreover, many problems lose their sense if the dependence on the past is not taken into account. All this leads to differential equations with delay of the argument [3–5]. Here we could also mention some recent works concerning the qualitative theory and numerical analysis of these equations and some generalizations of this notion [6–14].

A classical problem of the qualitative theory of differential equations is the existence of periodic solutions (for the case of differential equations with delay see [15–40], and also [41–45] where some applications to population dynamics are given, while for impulsive differential equations see the monograph [2] and the papers [44–47], quoted there as well as more recent works [48–52]). A traditional approach to this problem is the investigation of the linearized system (also called *system in variations*) with respect to a periodic solution of the unperturbed system satisfying certain non-degeneracy assumptions.

In the present paper for an impulsive system with a small delay it is proved that if the corresponding system without delay has an isolated ω-periodic solution, then in any sufficiently small neighbourhood of this orbit the system considered also has a unique ω-periodic solution. Thus the impulsive system without delay plays the role of an unperturbed system, and the delay that of a small perturbation. In the scalar (one-dimensional) case a similar problem was touched in [53, 54] using the method of successive approximations. In an earlier version of the present paper the result was proved under considerably more restrictive assumptions applying the contraction mapping principle [55, section 8]. Moreover, the result was extended to the case of a neutral impulsive system with a small delay [56] (see Remark 1 below for more details).

2. Statement of the problem. Main result

Consider the system with impulses at fixed moments

$$\begin{cases} \dot{x} = f(t, x(t), x(t-h)) & t \neq t_i \qquad t \neq t_i + h \\ \Delta x(t_i) = I_i(x(t_i), x(t_i - h)) & i \in \mathbb{Z} \\ \Delta x(t_i + h) = 0 \quad \text{if} \quad h > 0 \end{cases} \tag{1}$$

where $x \in \Omega \subset \mathbb{R}^n$, $f\colon \mathbb{R} \times \Omega \times \Omega \to \mathbb{R}^n$, $I_i : \Omega \times \Omega \to \mathbb{R}^n (i \in \mathbb{Z})$ are the impulses at moments t_i and $\{t_i\}_{i \in \mathbb{Z}}$ is a strictly increasing sequence such that

$$\lim_{i \to \pm\infty} t_i = \pm\infty$$

Ω is a domain in $\mathbb{R}^n$, $\Delta x(t_i) = x(t_i + 0) - x(t_i - 0)$, $h \geqslant 0$ is the delay.

As usual in the theory of impulsive differential equations, at the points of discontinuity t_i of the solution $x(t)$ we assume that $x(t_i) \equiv x(t_i - 0)$. It is clear that, in general, the derivatives $\dot{x}(t_i)$, $\dot{x}(t_i + h)$ do not exist. However, there exist the limits $\dot{x}(t_i \pm 0)$, $\dot{x}(t_i + h \pm 0)$. According to the above convention, we assume $\dot{x}(t_i) \equiv \dot{x}(t_i - 0)$, $\dot{x}(t_i + h) \equiv \dot{x}(t_i + h - 0)$.

For the sake of brevity we shall use the following notation:

$$\bar{x}(t) = x(t-h) \qquad x_i = x(t_i).$$

Introduce the following conditions:

H1. The function $f(t, x, \bar{x})$ is continuous, ω-periodic with respect to t, and continuously differentiable with respect to x, $\bar{x}$.

H2. The functions $I_i(x, \bar{x}) \in C^1(\Omega \times \Omega, \mathbb{R}^n)$, $i \in \mathbb{Z}$.

H3. There exists a positive integer m such that $t_{i+m} = t_i + \omega$, $I_{i+m}(x, \bar{x}) = I_i(x, \bar{x})$, $i \in \mathbb{Z}$, $x, \bar{x} \in \Omega$.

Together with (1) we consider the so called *generating* system

$$\begin{cases} \dot{x} = f(t, x(t), x(t)) & t \neq t_i \\ \Delta x(t_i) = I_i(x_i, x_i) & i \in \mathbb{Z} \end{cases} \tag{2}$$

obtained from (1) for $h = 0$, and suppose that;

H4. The generating system (2) has an ω-periodic solution $\psi(t)$ such that $\psi(t) \in \Omega$ for all $t \in \mathbb{R}$.

Next define the linearized system with respect to $\psi(t)$:

$$\begin{cases} \dot{y} = A(t)y & t \neq t_i \\ \Delta y(t_i) = B_i y_i & i \in \mathbb{Z} \end{cases} \tag{3}$$

where

$$A(t)=\frac{\partial}{\partial x}f(t,x,x)|_{x=\psi(t)} \qquad B_i=\frac{\partial}{\partial x}I_i(x,x)|_{x=\psi_i}. \tag{4}$$

Let $X(t)$ be the fundamental solution of (3) (i.e. $X(0)=E$, the unit matrix). Now we make the following additional assumptions:

H5. The matrix $E-X(\omega)$ is non-singular.

H6. The matrices $E+B_i$, $i\in\mathbb{Z}$, are non-singular.

The last two conditions allow us to define Green's function [2] of the periodic problem for a non-homogeneous system corresponding to (3) by the formula

$$G(t,\tau)=\begin{cases} X(t)(E-X(\omega))^{-1}X^{-1}(\tau) & 0\leqslant\tau<t\leqslant\omega \\ X(t+\omega)(E-X(\omega))^{-1}X^{-1}(\tau) & 0\leqslant t\leqslant\tau\leqslant\omega \end{cases}$$

and extend it as ω-periodic with respect to t and τ.

We may note the relation

$$G(t,t_i+0)=G(t,t_i)(E+B_i)^{-1}.$$

Our result in the present paper is the following:

Theorem 1. Let conditions H1–H6 hold. Then there exists a neighbourhood $\tilde{\Omega}\subset\Omega$ of the orbit $x=\psi(t)$ and a number $h_0>0$ such that for $h<h_0$ system (1) has in $\tilde{\Omega}$ a unique ω-periodic solution $x(t,h)$ such that $x(t,0)\equiv\psi(t)$.

Remark 1. As mentioned in the introduction, in the initial version of the paper the theorem was proved under stronger assumptions (Lipschitz continuity of the second derivatives in conditions H1 and H2) using the contraction mapping principle (see [55], section 8). In [56] this result was extended for the neutral system

$$\begin{cases} \dot{x}(t)=D(t)\dot{x}(t-h)+f(t,x(t),x(t-h)) & t\neq t_i+kh \quad i\in\mathbb{Z} \quad k\in\mathbb{N}\cup\{0\} \\ \Delta x(t_i)=I_i(x(t_i),x(t_i-h)),\ i\in\mathbb{Z} & \Delta x(t_i+kh)=0 \ \text{ if } \ t_i+kh\neq t_l \quad \forall l\in\mathbb{Z} \end{cases} \tag{4}$$

using again the contraction mapping principle under the additional assumption that the matrix $D(t)$ is ω-periodic, smooth and small enough. Obviously, the assumptions in this case can be weakened by applying the implicit function theorem [57] as below. We may note that in both cases under the same conditions (with just continuity of $D(t)$ in system (4)) we can apply the Schauder fixed point theorem instead, but then we cannot prove the local uniqueness of the periodic solution. The use of the implicit function theorem was kindly suggested by a referee.

3. Proof of the main result

3.1. Reduction of the problem to an operator equation in a suitable function space

There exists a constant $\delta_1>0$ such that Ω contains a closed δ_1-neighbourhood Ω_1 of the periodic orbit $x=\psi(t)$.

For a vector $x\in\mathbb{R}^n$ we denote by $|x|$ its Euclidean norm, and for an $n\times n$ matrix A we define the associated norm

$$|A|=\sup\{|Ax|/|x|\,;\,x\in\mathbb{R}^n\backslash 0\}.$$

In system (1) we change the variables according to the formula

$$x=y+\psi(t) \tag{5}$$

and obtain the system

$$\begin{cases}\dot{y}=A(t)y+Q(t,y)+\Delta f(t,y+\psi,\bar{y}+\bar{\psi}) & t\neq t_i \qquad t\neq t_i+h\\ \Delta y(t_i)=B_iy_i+J_i(y_i)+\Delta I_i(y_i+\psi_i,\bar{y}_i+\bar{\psi}_i) & i\in\mathbb{Z}\\ \Delta y(t_i+h)=0 \quad \text{for} \quad h>0, i\in\mathbb{Z}\end{cases} \tag{6}$$

where

$$\Delta f(t,x,\bar{x})\equiv f(t,x,\bar{x})-f(t,x,x)$$

$$\Delta I_i(x_i,\bar{x}_i)\equiv I_i(x_i,\bar{x}_i)-I_i(x_i,x_i)$$

are perturbations due to the delay while

$$Q(t,y)\equiv f(t,y+\psi(t),y+\psi(t))-f(t,\psi(t),\psi(t))-A(t)y \tag{7}$$

$$J_i(y)\equiv I_i(y+\psi_i,y+\psi_i)-I_i(\psi_i,\psi_i)-B_iy \tag{8}$$

are nonlinearities independent of the delay.

To find an ω-periodic solution of system (1) we shall find a solution $y(t)$ of system (6) satisfying $|y(t)|\leqslant\delta\leqslant\delta_1$. According to [2] the unique ω-periodic solution $z(t)$ of the linear system

$$\begin{cases}\dot{z}=A(t)z+Q(t,y)+\Delta f(t,y+\psi,\bar{y}+\bar{\psi}) & t\neq t_i\\ \Delta z(t_i)=B_iz_i+J_i(y_i)+\Delta I_i(y_i+\psi_i,\bar{y}_i+\bar{\psi}_i) & i\in\mathbb{Z}\\ \Delta z(t_i+h)=0 \quad \text{for} \quad h>0, i\in\mathbb{Z}\end{cases} \tag{9}$$

is given by the formula

$$z(t)=\int_0^\omega G(t,\tau)[Q(\tau,y(\tau))+\Delta f(\tau,y(\tau)+\psi(\tau),\bar{y}(\tau)+\bar{\psi}(\tau))]\,d\tau$$
$$+\sum_{0<t_i<\omega}G(t,t_i+0)[J_i(y_i)+\Delta I_i(y_i+\psi_i,\bar{y}_i+\bar{\psi}_i)] \tag{10}$$

where Green's function $G(t,\tau)$ was defined above.

Suppose, for the sake of definiteness, that

$$0<t_1<t_2<\ldots<t_m<\omega.$$

Let $h_1>0$ be so small that for any $h\in[0,h_1]$ we have

$$t_i+h<t_{i+1},\ i=\overline{1,m-1},\ t_m+h<\omega.$$

Denote by $\tilde{C}_{\omega,n}$ the space of all ω-periodic, piecewise continuous functions $w: \mathbb{R} \to \mathbb{R}^n$ with discontinuities of the first kind at t_i and $w(t_i - 0) = w(t_i)$, $i \in \mathbb{Z}$, supplied with the norm

$$\|w\| = \sup_{t \in \mathbb{R}} |w(t)|.$$

For $\delta \in (0, \delta_1]$ we define the subset

$$T_\delta = \{w \in \tilde{C}_{\omega,n}: \|w\| \leqslant \delta\}.$$

By $\tilde{C}^{(1)}_{\omega,n}$ we denote the space of all functions $w \in \tilde{C}_{\omega,n}$ having continuous derivatives on each interval of the form $(t_i, t_i + h)$ or $(t_i + h, t_{i+1})$, and such that there exist the limits $\dot{w}(t_i \pm 0)$ and $\dot{w}(t_i + h \pm 0)$. For such functions by definition we set $\dot{w}(t_i) = \dot{w}(t_i - 0)$, $\dot{w}(t_i + h) = \dot{w}(t_i + h - 0)$. For $\zeta > 0$ we define the subset

$$V_\zeta = \{w \in \tilde{C}^{(1)}_{\omega,n}: \sup_{t \in [0, \omega]} |\dot{w}(t)| \leqslant \zeta\}.$$

By formula (10) a solution of system (6) is a function $y \in \tilde{C}_{\omega,n}$ satisfying the equation

$$y = \mathscr{U}(h, y) \tag{11}$$

where the map $\mathscr{U}: [0, h_1] \times (T_\delta \subset \tilde{C}_{\omega,n}) \to \tilde{C}_{\omega,n}$ is defined as follows:

$$\mathscr{U}(h, y) = \mathscr{F}_1(y) + \mathscr{F}_2(h, y) + \mathscr{F}_3(y) + \mathscr{F}_4(h, y) \tag{12}$$

while

$$\mathscr{F}_1(y) = \int_0^\omega G(\cdot, \tau) Q(\tau, y(\tau))\, \mathrm{d}\tau,$$

$$\mathscr{F}_2(h, y) = \int_0^\omega G(\cdot, \tau) \Delta f(\tau, y(\tau) + \psi(\tau), \bar{y}(\tau) + \bar{\psi}(\tau))\, \mathrm{d}\tau,$$

$$\mathscr{F}_3(y) = \sum_{i=1}^m G(\cdot, t_i + 0) J(y_i)$$

$$\mathscr{F}_4(h, y) = \sum_{i=1}^m G(\cdot, t_i + 0) \Delta I_i(y_i + \psi_i, \bar{y}_i + \bar{\psi}_i).$$

3.2. Application of the implicit function theorem

We define the map $\mathscr{G}: [0, h_1] \times (T_\delta \subset \tilde{C}_{\omega,n}) \to \tilde{C}_{\omega,n}$ by

$$\mathscr{G}(h, y) = y - \mathscr{U}(h, y)$$

and seek a solution y of the equation

$$\mathscr{G}(h, y) = 0. \tag{13}$$

Obviously, $\mathscr{G}(0, 0) = 0$. We shall prove that:

(i) $\mathscr{G}$ is continuous with respect to h at the point $h = 0$ for y in T_{δ_1};

(ii) $\mathscr{G}$ is continuously differentiable with respect to y in the set $[0, h_1] \times T_{\delta_1}$, and $D_y \mathscr{G}(0, 0) = \mathrm{Id}$.

Assertion (i) means that $\mathscr{G}$ may be considered as a one-parameter family of perturbations of the map $\mathscr{G}_0 = \mathscr{G}(0, \cdot): T_\delta \to \tilde{C}_{\omega,n}$.

Following some arguments in [57], introduce the Banach spaces

$$\mathscr{B}_0 = C(T_\delta, \tilde{C}_{\omega,n}),\ \mathscr{B}_1 = \mathscr{B}_2 = \tilde{C}_{\omega,n}.$$

For $\mathscr{G}$ near $\mathscr{G}_0$ in $\mathscr{B}_0$ (i.e. h near 0) and y near 0 in $\mathscr{B}_1$ define

$$\Phi(\mathscr{G}, y) = \mathscr{G}(h, y) \in \mathscr{B}_2.$$

According to assertion (ii) Φ is continuously differentiable with respect to y for fixed $\mathscr{G}$, and the differentiability with respect to $\mathscr{G}$ is obvious since Φ is linear in $\mathscr{G}$. It follows (see exercise after Lemma 1.1 in [57]) that Φ is continuously differentiable in $(\mathscr{G}, y)$, and the differential for fixed $\mathscr{G}$ at $(\mathscr{G}, y) = (\mathscr{G}_0, 0)$ is the identity map in $\tilde{C}_{\omega,n}$.

Then by the implicit function theorem (Theorem 1.5 in [57]) there exists a C^1 map φ from a neighbourhood of $\mathscr{G}_0$ in $\mathscr{B}_0$ to a neighbourhood of 0 in $\mathscr{B}_1$ such that $y = \varphi(\mathscr{G})$ is the only solution close to 0 of $\Phi(\mathscr{G}, y) = 0$. In other words, there exists a number $h_0 \in (0, h_1]$ and a continuous map $y(h): [0, h_0) \to T_{\delta_0} \subset \tilde{C}_{\omega,n} (\delta_0 \in (0, \delta_1])$ satisfying equation (13) and $y(0) = 0$. This map, considered as a function of t depending on the delay h in a continuous way, gives the unique ω-periodic solution $y(t, h)$ of system (6) satisfying $\|y\| \leqslant \delta_0$. Then by formula (5) we obtain the unique ω-periodic solution $x(t, h)$ of system (1) in a δ_0-neighbourhood $\tilde{\Omega}$ of the periodic orbit $x = \psi(t)$. Moreover, $x(t, 0) = \psi(t)$.

3.3. *Proof of assertions (i) and (ii)*

First we introduce some notation. Let us denote

$$M_1 = \sup\{|f(t, x, \bar{x})| : t \in [0, \omega], x, \bar{x} \in \Omega_1\}$$

$$M_2 = \sup\{|f_x(t, x, \bar{x})| : t \in [0, \omega], x, \bar{x} \in \Omega_1\}$$

$$M_3 = \sup\{|f_{\bar{x}}(t, x, \bar{x})| : t \in [0, \omega], x, \bar{x} \in \Omega_1\}$$

$$M_4 = \sup\{|\partial_{\bar{x}} I_i(x, \bar{x})| : i = \overline{1, m}, x, \bar{x} \in \Omega_1\}$$

where $f_x, f_{\bar{x}}$ and $\partial_{\bar{x}} I_i$ are the partial derivatives of f with respect to x and $\bar{x}$, and of I_i with respect to $\bar{x}$, and

$$M = \sup\{|G(t, \tau)| : t, \tau \in [0, \omega]\}.$$

We introduce the following modules of continuity of the first derivatives of the function in condition H1:

$$\eta_1(\mu) = \sup\{|f_x(t, x, x+y) - f_x(t, x, x)| : t \in [0, \omega], x \in \Omega_1, \|y\| \leqslant \mu\}$$

$$\eta_2(\mu) = \sup\{|f_{\bar{x}}(t, x, x+y) - f_{\bar{x}}(t, x, x)| : t \in [0, \omega], x \in \Omega_1, \|y\| \leqslant \mu\}$$

$$\eta_3(\mu) = \sup\left\{\left|\frac{\partial}{\partial x}(f(t, x+y', x+y') - f(t, x+y'', x+y''))\right| :\right.$$
$$\left. t \in [0, \omega], x = \psi(t), |y'| \leqslant \delta_1, |y''| \leqslant \delta_1, |y' - y''| \leqslant \mu\right\}.$$

(i) It suffices to prove the continuity of $\mathscr{U}$ with respect to h at the point $h = 0$ for $y \in T_{\delta_1}$. From the representation (12) we find $\mathscr{U}(0, y) = \mathscr{F}_1(y) + \mathscr{F}_3(y)$. It remains to estimate $\mathscr{F}_2(h, y)$ and $\mathscr{F}_4(h, y)$ for y in a dense subset of T_δ, namely $y \in T_\delta \cap V_\zeta$ for some $\zeta > 0$.

Define the sets

$$\Delta_h = \bigcup_{i=1}^{m} (t_i, t_i + h) \qquad I_h = [0, \omega] \backslash \Delta_h .$$

For $t \in I_h$ the points t and $t-h$ belong to the same interval of continuity of the functions ψ and y, thus

$$|\bar{\psi}(t) - \psi(t)| = |\psi(t-h) - \psi(t)| \leqslant h \sup_{t \in (t-h,t)} |\dot{\psi}(t)| < hM_1$$

since ψ is a solution of (2), and

$$|\bar{y}(t) - y(t)| \leqslant h\zeta.$$

Now for $t \in I_h$ we have

$$\begin{aligned} &|\Delta f(t, \psi(t) + y, \bar{\psi}(t) + \bar{y})| \\ &\qquad = |f(t, \psi(t) + y, \bar{\psi}(t) + \bar{y}) - f(t, \psi(t) + y, \psi(t) + y)| \\ &\qquad \leqslant M_3 |\bar{y} - y + \bar{\psi}(t) - \psi(t)| \leqslant M_3 h(\zeta + M_1). \end{aligned}$$

For $t \in \Delta_h$ we use the rough estimate

$$|\Delta f(t, \psi(t) + y, \bar{\psi}(t) + \bar{y})| \leqslant 2M_1 .$$

This is sufficient for our aims since the measure of the set Δ_h is mh and it is small when h is small. Thus

$$\begin{aligned} |\mathscr{F}_2(h, y)(t)| &\leqslant \int_0^{\omega} |G(t, \tau)| |\Delta f(\tau, \psi(\tau) + y(\tau), \bar{\psi}(\tau) + \bar{y}(\tau))| \, d\tau \\ &= \int_{I_h} |G(t, \tau)| |\Delta f(\tau, \psi(\tau) + y(\tau), \bar{\psi}(\tau) + \bar{y}(\tau))| \, d\tau \\ &\quad + \sum_{i=1}^{m} \int_{t_i}^{t_i + h} |G(t, \tau)| |\Delta f(\tau, \psi(\tau) + y(\tau), \bar{\psi}(\tau) + \bar{y}(\tau))| \, d\tau \\ &\leqslant Mh[(\omega - mh) M_3 (\zeta + M_1) + 2mM_1]. \end{aligned} \tag{14}$$

Similarly we find

$$|\bar{\psi}_i - \psi_i| \leqslant hM_1 \qquad |\bar{y}_i - y_i| \leqslant h\zeta$$

$$|\Delta I_i(y_i + \psi_i, \bar{y}_i + \bar{\psi}_i)| \leqslant M_4 h(\zeta + M_1)$$

and finally

$$\|\mathscr{F}_4(h, y)\| \leqslant MmM_4 h(\zeta + M_1). \tag{15}$$

Estimates (14), (15) yield the desired continuity of the map $\mathscr{U}(h, \cdot)$ at the point $h = 0$.

(ii) Let us recall [57, 58] that the partial derivative $D_y \mathscr{G}(h, y)$ is a linear map $\tilde{C}_{\omega,n} \to \tilde{C}_{\omega,n}$ defined as follows:

$$D_y \mathscr{G}(h, y) z = \lim_{\mu \to 0} \mu^{-1} [\mathscr{G}(h, y + \mu z) - \mathscr{G}(h, y)].$$

Continuous differentiability of $\mathscr{G}(h, y)$ with respect to y means that the mapping $D_y\mathscr{G}(h, y)z: [0, h_1] \times (T_\delta \subset \tilde{C}_{\omega,n}) \times \tilde{C}_{\omega,n} \to \tilde{C}_{\omega,n}$ is continuous. Obviously we have

$$D_y\mathscr{G}(h, y) = \mathrm{Id} - D_y\mathscr{U}(h, y)$$

and

$$D_y\mathscr{U}(h, y) = D_y\mathscr{F}_1(y) + D_y\mathscr{F}_2(h, y) + D_y\mathscr{F}_3(y) + D_y\mathscr{F}_4(h, y) \tag{16}$$

provided that all derivatives on the right-hand sides exist. Thus it suffices to find that the derivatives on the right-hand side of (16) exist, are continuous in the above sense and vanish at $h=0$, $y=0$. We shall carry out the proof with all details only for the first addend.

$$\begin{aligned}
D_y\mathscr{F}_1(y)z &= \lim_{\mu\to 0} \mu^{-1}[\mathscr{F}_1(y+\mu z) - \mathscr{F}_1(y)] \\
&= \lim_{\mu\to 0} \mu^{-1} \int_0^\omega G(\cdot, \tau)[Q(\tau, y(\tau)+\mu z(\tau)) - Q(\tau, y(\tau))]\,\mathrm{d}\tau \\
&= \int_0^\omega G(\cdot, \tau) \lim_{\mu\to 0} \mu^{-1}[f(\tau, \psi(\tau)+y(\tau)+\mu z(\tau), \psi(\tau)+y(\tau)+\mu z(\tau)) \\
&\quad - f(\tau, \psi(\tau)+y(\tau), \psi(\tau)+y(\tau)) - \mu A(\tau)z(\tau)]\,\mathrm{d}\tau \\
&= \int_0^\omega G(\cdot, \tau)\left(\frac{\partial}{\partial x} f(\tau, x, x)|_{x=\psi(\tau)+y(\tau)} - \frac{\partial}{\partial x} f(\tau, x, x)|_{x=\psi(\tau)}\right) z(\tau)\,\mathrm{d}\tau.
\end{aligned}$$

Obviously $D_y\mathscr{F}_1(0)=0$. To prove the continuity of the derivative we take $y', y'', z', z'' \in \tilde{C}_{\omega,n}$, $\|y'\| \leqslant \delta$, $\|y''\| \leqslant \delta$. Then we have

$$\begin{aligned}
&|(D_y\mathscr{F}_1(y')z' - D_y\mathscr{F}_1(y'')z'')(t)| \\
&\quad\leqslant \left|\int_0^\omega G(t, \tau)\left(\frac{\partial}{\partial x} f(\tau, x, x)|_{x=\psi(\tau)+y'(\tau)} \right.\right. \\
&\qquad \left.\left. - \frac{\partial}{\partial x} f(\tau, x, x)|_{x=\psi(\tau)+y''(\tau)}\right) z'(\tau)\,\mathrm{d}\tau\right| \\
&\qquad + \left|\int_0^\omega G(t, \tau)\left(\frac{\partial}{\partial x} f(\tau, x, x)|_{x=\psi(\tau)+y''(\tau)} \right.\right. \\
&\qquad \left.\left. - \frac{\partial}{\partial x} f(\tau, x, x)|_{x=\psi(\tau)}\right)(z'(\tau) - z''(\tau))\,\mathrm{d}\tau\right| \\
&\quad\leqslant M\omega(\eta_3(\|y'-y''\|)\|z'\| + \eta_3(\delta)\|z'-z''\|)
\end{aligned}$$

which yields the desired continuity.

Analogously we obtain

$$D_y\mathscr{F}_3(y)z = \sum_{i=1}^m G(\cdot, t_i+0)\left(\frac{\partial}{\partial x} I_i(x, x)|_{x=\psi_i+y_i} - \frac{\partial}{\partial x} I_i(x, x)|_{x=\psi_i}\right) z_i$$

and $D_y\mathscr{F}_3(0)=0$. The continuity of the derivative is almost obvious in view of condition H2.

Next we find

$$D_y\mathscr{F}_2(h,y)z=\int_0^\omega G(\cdot,\tau)\Bigg\{\Bigg[f_x(\tau,\psi(\tau)+y(\tau),\bar{\psi}(\tau)+\bar{y}(\tau))z(\tau)$$

$$+f_{\bar{x}}(\tau,\psi(\tau)+y(\tau),\bar{\psi}(\tau)+\bar{y}(\tau))\bar{z}(\tau)\Bigg]$$

$$-\Bigg[f_x(\tau,\psi(\tau)+y(\tau),\psi(\tau)+y(\tau))$$

$$+f_{\bar{x}}(\tau,\psi(\tau)+y(\tau),\psi(\tau)+y(\tau))\Bigg]z(\tau)\Bigg\}\,\mathrm{d}\tau.$$

Obviously we have $D_y\mathscr{F}_2(0,y)=0$. The above arguments show that it suffices to prove the continuity of the derivative with respect to h. Here we must consider two cases:

(a) continuity at $h=0$. It suffices to take $y\in T_\delta\cap V_\zeta$, $z\in V_\zeta$. We use the representation

$$D_y\mathscr{F}_2(h,y)z=\int_0^\omega G(\cdot,\tau)\Bigg\{\Bigg[f_x(\tau,\psi(\tau)+y(\tau),\bar{\psi}(\tau)+\bar{y}(\tau))$$

$$-f_x(\tau,\psi(\tau)+y(\tau),\psi(\tau)+y(\tau))\Bigg]z(\tau)$$

$$+\Bigg[f_{\bar{x}}(\tau,\psi(\tau)+y(\tau),\bar{\psi}(\tau)+\bar{y}(\tau))\bar{z}(\tau)$$

$$-f_{\bar{x}}(\tau,\psi(\tau)+y(\tau),\psi(\tau)+y(\tau))z(\tau)\Bigg]\Bigg\}\,\mathrm{d}\tau.$$

For $\tau\in I_h$ the points $\tau-h$ and τ belong to the same interval of continuity of the functions ψ, y and z. Thus we have

$$|\bar{\psi}(\tau)-\psi(\tau)|\leqslant hM_1 \qquad |\bar{y}(\tau)-y(\tau)|\leqslant\zeta h \qquad |\bar{z}(\tau)-z(\tau)|\leqslant\zeta h$$

$$|f_x(\tau,\psi(\tau)+y(\tau),\bar{\psi}(\tau)+\bar{y}(\tau))-f_x(\tau,\psi(\tau)+y(\tau),\psi(\tau)+y(\tau))|\leqslant\eta_1(h(M_1+\zeta))$$

$$\begin{aligned}&|f_{\bar{x}}(\tau,\psi(\tau)+y(\tau),\bar{\psi}(\tau)+\bar{y}(\tau))\bar{z}(\tau)-f_{\bar{x}}(\tau,\psi(\tau)+y(\tau),\psi(\tau)+y(\tau))z(\tau)|\\&\quad\leqslant|f_{\bar{x}}(\tau,\psi(\tau)+y(\tau),\bar{\psi}(\tau)+\bar{y}(\tau))-f_{\bar{x}}(\tau,\psi(\tau)+y(\tau),\psi(\tau)+y(\tau))\|\bar{z}(\tau)|\\&\qquad+|f_{\bar{x}}(\tau,\psi(\tau)+y(\tau),\psi(\tau)+y(t))\|\bar{z}(\tau)-z(\tau)|\\&\quad\leqslant\eta_2(h(M_1+\zeta))\|z\|+M_3h\zeta.\end{aligned}$$

On the set Δ_h whose measure is mh we again use the rough estimates

$$|f_x(\tau,\psi(\tau)+y(\tau),\bar{\psi}(\tau)+\bar{y}(\tau))-f_x(\tau,\psi(\tau)+y(\tau),\psi(\tau)+y(\tau))\|z(\tau)|\leqslant 2M_2\|z\|$$

$$|f_{\bar{x}}(\tau,\psi(\tau)+y(\tau),\bar{\psi}(\tau)+\bar{y}(\tau))\bar{z}(\tau)-f_{\bar{x}}(t,\psi(\tau)+y(\tau),\psi(\tau)+y(\tau))z(\tau)|\leqslant 2M_3\|z\|$$

and obtain

$$\|D_y\mathscr{F}_2(h,y)z\| \leqslant M\{(\omega-mh)[[\eta_1(h(M_1+\zeta))+\eta_2(h(M_1+\zeta))]\|z\|+M_3h\zeta] \\ +2mh(M_2+M_3)\|z\|\}$$

which tends to 0 as $h\to 0$.

(b) To prove the continuity of $D_y\mathscr{F}_2(h,y)z$ at some $h>0$ it suffices to consider just the two terms

$$\int_0^\omega G(\cdot,\tau)f_x(\tau,\psi(\tau)+y(\tau),\bar{\psi}(\tau)+\bar{y}(\tau))z(\tau)\,\mathrm{d}\tau \tag{17}$$

and

$$\int_0^\omega G(\cdot,\tau)f_{\bar{x}}(\tau,\psi(\tau)+y(\tau),\bar{\psi}(\tau)+\bar{y}(\tau))\bar{z}(\tau)\,\mathrm{d}\tau. \tag{18}$$

We shall consider only the first of these. Take h', h'' such that $0<h'<h''\leqslant h_1$ and estimate the difference

$$\int_0^\omega G(t,\tau)[f_x(\tau,\psi(\tau)+y(\tau),\psi(\tau-h') \\ +y(\tau-h'))-f_x(\tau,\psi(\tau)+y(\tau),\psi(\tau-h'')+y(\tau-h''))]\,z(\tau)\,\mathrm{d}\tau$$

for $y\in T_\delta\cap V_\zeta$. *If* $\tau\notin(t_i,t_i+h'')$ for all i or $\tau\in(t_i,t_i+h')$ for some i, the points $\tau-h'$ and $\tau-h''$ belong to the same interval of continuity of the functions ψ and y, thus

$$|\psi(\tau-h')-\psi(\tau-h'')|\leqslant M_1(h''-h')$$
$$|y(\tau-h')-y(\tau-h'')|\leqslant\zeta(h''-h').$$

Thus on a set with measure $\omega-m(h''-h')$ we have the estimate

$$|f_x(\tau,\psi(\tau)+y(\tau),\psi(\tau-h')+y(\tau-h'))-f_x(\tau,\psi(\tau)+y(\tau),\psi(\tau-h'')+y(\tau-h''))| \\ \leqslant\eta_1((M_1+\zeta)(h''-h')).$$

On the other hand, if $\tau\in(t_i+h',t_i+h'')$, we use the rough estimate

$$|f_x(\tau,\psi(\tau)+y(\tau),\psi(\tau-h')+y(\tau-h'))-f_x(\tau,\psi(\tau)+y(\tau),\psi(\tau-h'')+y(\tau-h''))| \\ \leqslant 2M_2.$$

As above, we obtain the inequality

$$\left|\int_0^\omega G(t,\tau)[f_x(\tau,\psi(\tau)+y(\tau),\psi(\tau-h')+y(\tau-h')) \\ -f_x(\tau,\psi(\tau)+y(\tau),\psi(\tau-h'')+y(\tau-h''))]z(\tau)\,\mathrm{d}\tau\right| \\ \leqslant M\{[\omega-m(h''-h')]\eta_1((M_1+\zeta)(h''-h'))+2M_2m(h''-h')\}.$$

The last estimate shows the continuity of the integral (17) with respect to $h>0$. Similar arguments are used for the integral (18).

Finally we obtain

$$D_y\mathscr{F}_4(h, y)z = \sum_{i=1}^{m} G(\cdot, t_i+0)[\partial_x I_i(\psi_i+y_i, \bar{\psi}_i+\bar{y}_i)z_i + \partial_{\bar{x}} I_i(\psi_i+y_i, \bar{\psi}_i+\bar{y}_i)\bar{z}_i$$

$$-(\partial_x I_i(\psi_i+y_i, \psi_i+y_i) + \partial_{\bar{x}} I_i(\psi_i+y_i, \psi_i+y_i))z_i].$$

It is easily seen that this derivative is continuous with respect to h, y, z since the functions y, z, ψ are continuous in the intervals $[t_i-h, t_i]$ and that $D_y\mathscr{F}_4(0, y)z=0$.

Thus we have shown that assertion (ii) is valid and the arguments from section 3.2 complete the proof of the theorem. □

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References

[1] Lakshmikantham V, Bainov D and Simeonov P 1989 *Theory of Impulsive Differential Equations* (Singapore: World Scientific)
[2] Samoilenko A M and Perestyuk N A 1987 *Differential Equations with Impulse Effect* (Kiev: Višča Škola) (in Russian)
[3] Bellman R and Cooke K 1963 *Differential Difference Equations* (New York: Academic)
[4] Hale J K 1977 *Theory of Functional Differential Equations* (Berlin: Springer)
[5] Kuang Y 1993 *Delay Differential Equations with Applications in Population Dynamics* (New York: Academic)
[6] Buhmann M D and Iserles A 1992 On the dynamics of a discretized neutral equation *IMA J. Num. Anal.* **12** 339–63
[7] Buhmann M D and Iserles A 1992 Numerical analysis of functional equations with a variable delay *Numerical Analysis, Dundee, 1991* ed D F Griffiths and G A Watson Pitman Res. Notes Math. Ser. **260** (Harlow: Longman) pp 17–33
[8] Buhmann M D and Iserles A 1993 Stability of the discretized pantograph differential equation *Maths. Comp.* **60** 575–89
[9] Feldstein A, Iserles A and Levin D 1994 Embedding of delay equations into an infinite-dimensional ODE system *J. Diff. Equ.* (to appear)
[10] Iserles A 1993 On the generalized pantograph functional-differential equation *Euro. J. Appl. Math.* **4** 1–38
[11] Iserles A 1994 On nonlinear delay differential equations *Trans. Amer. Math. Soc.* (to appear)
[12] Iserles A 1994 Numerical analysis of delay differential equations *Ann. Num. Maths.* (to appear)
[13] Iserles A and Liu Y 1994 On neutral functional-differential equations with proportional delays (submitted to *SIAM J. Math. Anal.*)
[14] Iserles A and Terjéki J 1994 Stability and asymptotic stability of functional-differential equations *J. London Math. Soc.* (to appear)
[15] Chen Y S The existence of periodic solutions of the equation $x'(t) = -f(x(t), x(t-\tau))$ *J. Math. Anal. Applic.* **163** 227–37
[16] Chow S 1974 Existence of periodic solutions of autonomous functional differential equations *J. Diff. Equ.* **15** 350–78
[17] Grafton R 1969 A periodicity theorem for autonomous functional differential equations *J. Diff. Equ.* **6** 87–109
[18] Grafton R 1972 Periodic solutions of certain Liénard equations with delay *J. Diff. Equ.* **11** 519–27
[19] Grafton R 1972 Periodic solutions of Liénard equations with delay: some theoretical and experimental results *Delay and Functional Differential Equations and Their Applications* ed K Schmitt (New York: Academic) pp 321–34

[20] Hale J K 1970 Solutions near simple periodic orbits of functional differential equations, *J. Diff. Equ.* **9** 126–38
[21] Jones G 1962 The existence of periodic solutions of $f'(x) = -af(x-1).[1+f(x)]$ *J. Math. Anal. Applic.* **5** 435–50
[22] Jones G 1964 Periodic motions in Banach space and applications to functional differential equations, *Contributions to Diff. Equ.* **3** 75–106
[23] Kaplan and Yorke J 1974 Ordinary differential equations which yield periodic solutions of differential delay equations *J. Math. Anal. Applic.* **48** 317–25
[24] Lillo J C 1966 Periodic differential difference equations *J.Math. Anal. Applic.* **15** 434–41
[25] Lillo J C 1968 The Green's function for periodic differential difference equations *J. Diff. Equ.* **4** 373–85
[26] Lopes O 1974 Periodic solutions of perturbed neutral differential equations *J.Diff. Equa.* **15** 70–6
[27] Mallet-Paret J 1977 Generic periodic solutions of functional differential equations, *J. Diff. Equ.* **25** 163–83
[28] Mawhin J 1971 Periodic solutions of nonlinear functional differential equations *J. Diff. Equ.* **10** 240–61
[29] Nosov V R 1971 Periodic solutions of systems of linear equations of general type with delay of the argument *Diff. Equ.* **7** 639–50 (in Russian)
[30] Nosov V R 1973 Periodic solutions of quasilinear functional differential equations *Izv. Vuzov. Ser. Math.* **132** 55–62 (in Russian)
[31] Nussbaum R 1974 Periodic solutions of some nonlinear autonomous functional differential equations *Ann. Mat. Pura Appl.* **10** 263–306
[32] Nussbaum R 1973 Periodic solutions of some nonlinear autonomous functional differential equations II, *J. Diff. Equ.* **14** 368–94
[33] Perelló C 1968 Periodic solutions of differential equations with time lag containing a small parameter *J. Diff. Equ.* **4** 160–75
[34] Perelló C 1967 A note on periodic solutions of nonlinear differential equations with time lag in *Differential Equations and Dynamical Systems* (New York: Academic) pp 185–7
[35] Serra E 1991 Periodic solutions for some nonlinear differential equations of neutral type *Nonlinear Anal.* **17** 139–51
[36] Stephan B H 1970 Periodic solutions of differential equations with almost constant time lag *J. Diff. Equ.* **8** 554–63
[37] Wang K 1990 Periodic solutions of some differential difference equations *J. Math.* (*Uhan*) **10** 161–70
[38] Wang K 1990 On the existence of nontrivial periodic solutions of differential difference equations *Chinese Ann. Math. Ser.* B **11** 438–44
[39] Wexler D 1967 Solutions périodiques des systèmes linéaires à argument retardé *J. Diff. Equ.* **3** 336–47
[40] Ziegler H J 1968 Sur une note de M. C. Perelló sur des solutions périodiques d'équations fonctionnelles différentielles *C.R. Acad Sci. Paris* Ser. A-B **267** A783–5
[41] Swick K E 1980 Periodic solutions of a nonlinear age-dependent model of single species population dynamics *SIAM J. Math. Anal.* **11** 901–10
[42] Walther H 1975 Existence of a nonconstant periodic solution of a nonlinear nonautonomous functional differential equation representing the growth of a single species population *J. Math. Bio.* **1** 227–40
[43] Yuan J 1988 Periodic solutions of a nonlinear autonomous age-dependent model of single species population dynamics *Nonlinear Analysis* **12** 1079–86
[44] Perestyuk N A and Shovkoplyas V N 1979 Periodic solutions of nonlinear differential equations with impulse effect *Ukr. Math. J.* **31** 517–24 (in Russian)
[45] Samoilenko A M and Perestyuk N A 1978 Periodic solutions of weakly nonlinear systems with impulse effect *Diff. Equ.* **14** 1034–45 (in Russian)
[46] Samoilenko A M and Perestyuk N A 1982 Periodic and almost periodic solutions of differential equations with impulse effect *Ukr. Math. J.* **34** 66–73 (in Russian)
[47] Tsidylo K V and Gul'ka S S 1981 On the periodic solutions of nonlinear systems with impulse effect *Dokl. Acad. Sci. Ukr. SSR*, Ser. A 21–3 (in Russian)
[48] Ahmetov M U and Perestyuk N A 1991 Periodic solutions of quasilinear impulse systems in the critical case *Ukr. Math. J.* **43** 273–9
[49] Boichuk A A, Perestyuk N A and Samoilenko A M 1991 Periodic solutions of impulse differential systems in critical cases *Diff. Equ.* **27** 1070–73
[50] Erbe L H and Liu X 1991 Existence of periodic solutions of impulsive differential systems *J. Appl. Math. Stochastic Anal.* **4** 137–46

[51] Liz E and Nieto J J 1991 Periodic solutions of discontinuous impulsive differential systems *J. Math. Anal. Applic.* **161** 388–94

[52] Simeonov P S and Bainov D D 1991 Application of the method of the two-sided approximations to the solution of the periodic problem for impulsive differential equations *Tamkang J. Math* **22** 275–84

[53] Hristova S G and Bainov D D 1987 The method of the small parameter in the theory of differential-difference equations with impulse effect *Math. Rep. Toyama Univ.* **10** 23–36

[54] Hristova S G and Bainov D D 1987 The method of the small parameter in the theory of differential equations with impulse effect *Riv. Mat. Univ. Parma* **13** 91–100

[55] Bainov D and Covachev V 1994 *Impulsive Differential Equations with a Small Parameter* (Singapore: World Scientific)

[56] Jódar L, Villanueva R J and Covachev V Chr 1993 Periodic solutions of neutral impulsive systems with a small delay in *Proc. 4th Int. Colloquium on Differential Equations. (Plovdiv, Bulgaria, 1993)* (Utrecht)

[57] Hörmander L 1977 *Implicit Function Theorems* Lecture Notes, Stanford University

[58] Hamilton R S 1982 The inverse function theorem of Nash and Moser *Bull (New Series) Amer. Math. Soc.* **7** 65–222